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The Uncertainty of Pulse Position Due to Noise

DON J. TORRIERI

Space Applications Branch
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THE UNCERTAINTY OF PULSE POSITION DUE TO NOISE

FLUCTUATIONS IN THE PRESENCE OF GAUSSIAN NOISE

The time at which a received signal crosses a certain level fluctuates in the presence of noise. In this section theoretical formulas for the probability density of the crossing time and its standard deviation are obtained. It is assumed that both the signal and noise have been modified by the presence of an intermediate frequency (IF) filter, as shown in Fig. 1 where $s(t)$ is the signal and $n(t)$ is the noise after demodulation. It can be shown that the effect of the IF filter passband on the signal and noise is the same as the effect of a baseband filter of the same shape. Hence, in the following analysis, the IF filter will always be treated as if it were a baseband filter. Fig. 2 shows the signal and the threshold level a which must be crossed to trigger an output signal from the threshold detector. If an interval of time ϵ is chosen small enough and if both $s(t)$ and $n(t)$ are differentiable functions, then

and
$$\left.\begin{aligned} s(t_0+\epsilon) &= s(t_0) + s'(t_0)\epsilon \\ n(t_0+\epsilon) &= n(t_0) + n'(t_0)\epsilon, \end{aligned}\right\} \qquad (1)$$

where $s'(t_0)$ and $n'(t_0)$ are time derivatives at the point $t = t_0$. The probability P that $s(t) + n(t)$ crosses the threshold level during the interval ϵ shall now be calculated.

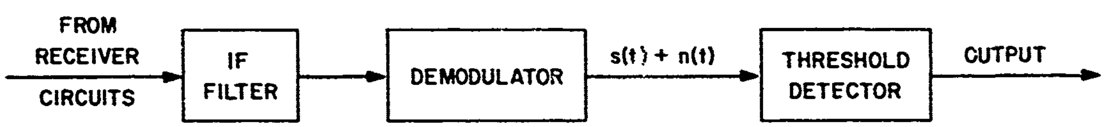

Fig. 1—Pulse position detection system

For a fixed value of $n'(t_0)$, we must have $s(t_0) + n(t_0) < a$ and $s(t_0 + \epsilon) + n(t_0 + \epsilon) > a$. Using Eq. (1), these two inequalities become $a - s(t_0) - s'(t_0)\epsilon - n'(t_0)\epsilon < n(t_0) < a - s(t_0)$. This inequality can be satisfied only if $n'(t_0) > - s'(t_0)$. With the aid of these last two inequalities, it is seen that

$$P = \int_{-M}^{\infty} \int_{a-s(t_0)-M\epsilon-\beta\epsilon}^{a-s(t_0)} f_{nn'}(\alpha, \beta)\, d\alpha\, d\beta, \qquad (2)$$

where $M = s'(t_0)$ for notational conveneince and $f_{nn'}(\alpha, \beta)$ is the joint probability density function of the noise and its derivative. It is now assumed that $n(t)$ is a stationary Gaussian process with a probability density function given by

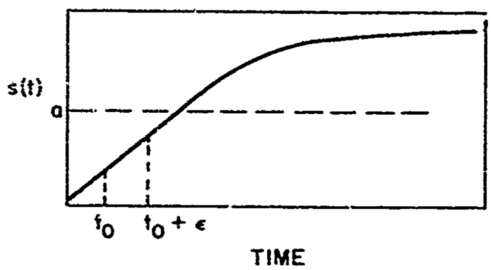

Fig. 2—Demodulator output signal

$$f_n(\alpha) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left[-\frac{(\alpha - \overline{n})^2}{2\sigma_n^2}\right], \tag{3}$$

where $\overline{n}$ is the mean (expected) value of $n(t)$ and σ_n^2 is the mean square of $n(t)$. Since $n'(t)$ can be considered the result of passing $n(t)$ through a linear system, $n'(t)$ is also a stationary Gaussian process. For a linear system with a system function $H(\omega)$ and a stationary random input, it is easy to show that

$$\overline{y} = H(0)\overline{x}, \tag{4}$$

where $\overline{y}$ is the mean value of the output and $\overline{x}$ is the mean value of the input. The system function of a differentiator is $H(\omega) = j\omega$. Thus it follows from Eq. (4) that $\overline{n}' = 0$. and we have the following expression for the probability density function of $n'(t)$:

$$f_{n'}(\beta) = \frac{1}{\sqrt{2\pi}\,\sigma_{n'}} \exp\left[-\frac{\beta^2}{2\sigma_{n'}^2}\right], \tag{5}$$

where $\sigma_{n'}^2$ is the mean square of $n'(t)$.

It shall now be shown that $n(t)$ and $n'(t)$ are uncorrelated. For a stationary process, we have the general relation

$$\overline{n(t_1)n'(t_2)} = -\frac{d}{d\tau}R(\tau), \tag{6}$$

where $\tau = t_1 - t_2$ and $R(\tau)$ is the autocorrelation function. Because $R(\tau)$ is an even function for a real process $n(t)$, the derivative in Eq. (6) vanishes at the origin if it exists there. Thus

$$\overline{n(t)n'(t)} = -\frac{d}{d\tau}R(\tau)\bigg|_{\tau=0} = 0, \tag{7}$$

and $n(t)$ has been shown to be uncorrelated with $n'(t)$.

Since $n(t)$ and $n'(t)$ may be considered as the outputs of two linear systems with the same input $n(t)$, it follows that $n(t)$ and $n'(t)$ are jointly normal. Since these two processes are also uncorrelated, they are independent; thus Eq. (2) becomes

$$P = \int_{-M}^{\infty} f_{n'}(\beta)\, d\beta \int_{a-s(t_0)-M\epsilon-\beta\epsilon}^{a-s(t_0)} f_n(\alpha)\, d\alpha. \tag{8}$$

Since ϵ may be chosen arbitrarily small, Eq. (8) reduces to

$$P = \epsilon f_n[a - s(t_0)] \int_{-M}^{\infty} (M + \beta) f_{n'}(\beta)\, d\beta. \tag{9}$$

The integral may be evaluated by substitution of Eq. (5) and a change of variables. The result is

$$P = \epsilon f_n[a - s(t)] \left[M \operatorname{erf}\left(\frac{M}{\sigma_{n'}}\right) + \frac{\sigma_{n'}}{\sqrt{2\pi}} \exp\left(-\frac{M^2}{2\sigma_{n'}^2}\right)\right], \tag{10}$$

where the subscript on the time variables has been dropped and the time dependence of M is suppressed for notational brevity. We have also defined

$$\operatorname{erf}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{x^2}{2}\right) dx. \tag{11}$$

At this point, it is tempting to define the probability density function for a threshold-level crossing at time t by $f_c(t) = P/\epsilon$ in the limit of small ϵ. Such a definition gives us, however, a density function that cannot be normalized. This anomaly is due to the fact that there may be more than one threshold-level crossing. In fact, there may be more than one crossing even when there is no signal present at all.

Crossings that do not occur near the time at which the signal level reaches the threshold level constitute a "false alarm" problem. This problem will be discussed in the next section. For the present, it is assumed that these distant crossings have been suitably excluded. It is further assumed that there exists an interval $t_1 < t < t_2$ in which there is a negligible probability of more than one threshold crossing and unity probability of exactly one crossing. We then can define a normalized probability density by $f_c(t) = P/\epsilon$ in this interval and $f_c(t) = 0$ otherwise. From Eq. (10), we have

$$\begin{aligned} f_c(t) &= f_n[a - s(t)] \left\{ M \operatorname{erf}\left(\frac{M}{\sigma_{n'}}\right) + \frac{\sigma_{n'}}{\sqrt{2\pi}} \exp\left(-\frac{M^2}{2\sigma_{n'}^2}\right)\right\}, \quad t_1 < t < t_2, \\ f_c(t) &= 0, \qquad t < t_1,\ t > t_2, \end{aligned} \tag{12}$$

where t_1 and t_2 must be such that normalization is possible; that is,

$$\int_{t_1}^{t_2} f_c(t)\, dt = 1. \tag{13}$$

It is immediately seen that t_1 and t_2 are not uniquely defined by Eq. (13). To remedy this defect, we first define the mean threshold time by

$$\bar{t} = \int_{t_1}^{t_2} t f_c(t)\, dt. \tag{14}$$

We then require that the probability of threshold crossing before $\bar{t}$ be equal to the probability of threshold crossing after $\bar{t}$; that is, we require

$$\int_{t_1}^{\bar{t}} f_c(t)\, dt = \int_{\bar{t}}^{t_2} f_c(t)\, dt. \tag{15}$$

The values of t_1 and t_2 may be determined by solving Eqs. (13)-(15) simultaneously.

A special case arises when $\bar{n} = 0$, $s(t)$ is antisymmetric about $\bar{t}$, and $\bar{t}$ is the same time as that time when $s(t)$ would cross the threshold level in the absence of noise; that is, $s(\bar{t}) = a$. Such a situation is shown in Fig 2. In this case $f_c(t)$ is symmetric about $\bar{t}$, and Eq. (15) requires that $t_1 = \bar{t} - T$ and $t_2 = \bar{t} + T$. The parameter T is determined by substitution in Eq. (13). We have

$$\int_{\bar{t}-T}^{\bar{t}+T} f_c(t)\, dt = 1, \tag{16}$$

which may be solved to determine T and, consequently, t_1 and t_2. These results will be referred to later.

There remains the task of justifying the assumption that led to Eq. (12). We assumed that there is negligible probability of more than one threshold crossing in the interval $t_1 < t < t_2$. If there is more than one threshold crossing from below threshold to above it, then there must be at least one threshold crossing from above threshold to below it. Thus the probability of multiple threshold crossing in an interval will be negligible if the probability of crossing from above the threshold to below it is negligible. Let P_R be the probability of this reversed threshold crossing in an interval between time t and $t + \epsilon$. Reasoning as in Eq. (2), it follows that

$$P_R = \int_{-\infty}^{-M} \int_{a-s(t)}^{a-s(t)-M\epsilon-\beta\epsilon} f_{nn'}(\alpha, \beta)\, d\alpha\, d\beta. \tag{17}$$

Calculating in a manner analogous to that leading to Eq. (12), we obtain the probability density function of a reverse crossing in the interval $t_1 < t < t_2$:

$$f_r(t) = f_n[a - s(t)] \left[-M \operatorname{erf}\left(-\frac{M}{\sigma_{n'}}\right) + \frac{\sigma_{n'}}{\sqrt{2\pi}} \exp\left(-\frac{M^2}{2\sigma_{n'}^2}\right)\right]. \tag{18}$$

If

$$0 \leqslant \int_{t_1}^{t_2} f_r(t)\, dt \ll 1, \tag{19}$$

then reverse threshold crossings and multiple crossings are of negligible importance. In this case, Eq. (12) is a mathematically consistent and accurate expression for the threshold-crossing density function in the vicinity of the pulse edge. Consequently, we may define the standard deviation σ_c of the threshold crossing time by

$$\sigma_c^2 = \int_{t_1}^{t_2} (t - \bar{t})^2 f_c(t)\, dt. \tag{20}$$

The implications of Eq. (19) shall now be elaborated. From Eqs. (12) and (13), we have

$$\int_{t_1}^{t_2} \frac{\sigma_{n'}}{\sqrt{2\pi}} \exp\left(-\frac{M^2}{2\sigma_{n'}^2}\right) f_n[a - s(t)]\, dt = 1 - \int_{t_1}^{t_2} M \operatorname{erf}\left(\frac{M}{\sigma_{n'}}\right) f_n[a - s(t)]\, dt \tag{21}$$

Substitution of Eqs. (18) and (21) into Eq. (19) yields, after some manipulation,

$$0 \leqslant 1 - \int_{t_1}^{t_2} Mf_n[a - s(t)]\, dt \ll 1. \tag{22}$$

We conclude that Eq. (22) must be satisfied for Eqs. (12) and (20) to be applicable. It is noteworthy that Eq. (22) implies that $M > 0$ in at least a subinterval of the interval $t_1 < t < t_2$. In particular, if M is a constant, it must be positive.

A special case of interest occurs when the noise is so weak that there is essentially zero probability of more than one threshold crossing in the interval $t_1 < t < t_2$. Since $f_r(t) = 0$, it follows that

$$M \operatorname{erf}\left(\frac{M}{\sigma_n'}\right) = \frac{\sigma_n'}{\sqrt{2\pi}} \exp\left(-\frac{M^2}{2\sigma_{n'}^2}\right) \tag{23}$$

The use of this relation in Eq. (12) then gives

$$\left.\begin{aligned} f_c(t) &= Mf_n[a - s(t)], && t_1 < t < t_2, \\ f_c(t) &= 0, && t < t_1, t > t_2. \end{aligned}\right\} \tag{24}$$

It should be remembered that Eq. (12) is valid for any stationary Gaussian noise process, including nonwhite Gaussian noise. The formula does not apply to non-Gaussian noise. However, it is shown in the appendix that Eq. (24) is approximately valid for weak noise, even if it is non-Gaussian.

FALSE ALARMS

If the noise entering a pulse-detecting system has an amplitude exceeding the threshold value, a spurious detection may occur. This phenomenon is called a false alarm. To reduce the probability of a false alarm, a detector should include a rejector circuit which blocks all input pulses which do not exceed a fixed level for a specified period of time.

Let L be the rejection level and T the time duration which must be exceeded if the pulse is to be passed by the rejector. We now determine the probability that a noise pulse will have an amplitude greater than L for the time T_1. This probability may be deduced by the same reasoning that led to Eq. (8). Thus if $P_N(T_1)$ is the probability of a noise pulse of duration T_1, then

$$P_N(T_1) = \int_0^\infty f_n'(\beta)\, d\beta \int_L^\infty f_n(\alpha)\, d\alpha + \int_{-\infty}^0 f_n'(\beta)\, d\beta \int_{L-\beta T_1}^\infty f_n(\alpha)\, d\alpha. \tag{25}$$

This relation gives the probability of a noise pulse occurring during an arbitrary interval of duration T_1. It is an approximate relation which increases in accuracy as T_1 decreases. For larger T_1, the expansion of Eq. (1) is no longer valid and, consequently, neither is Eq. (25).

Using Eqs. (3) and (5) and assuming $\bar{n} = 0$, Eq. (25) reduces to

$$P_N(T_1) = 1 - \frac{1}{2}\,\mathrm{erf}\left(\frac{L}{\sigma_n}\right) - \int_0^\infty f_{n'}(\beta)\,\mathrm{erf}\left(\frac{L + \beta T_1}{\sigma_n}\right) d\beta. \tag{26}$$

Suppose a pulse of duration $T_1 > T$ passes the rejector. The probability that a noise pulse of duration T_1 occurred during the same interval and that no signal was received during this interval is given by the joint probability function $P_{N,\phi}$. Let $P_{\phi|N}$ be the conditional probability that no signal was received, given that a noise pulse occurred. If P_F is the probability that the passed pulse is a false alarm, then

$$P_F = P_{N,\phi} = P_N P_{\phi|N} = P_N P_\phi, \tag{27}$$

where the last equality follows from the fact that the signal and noise are statistically independent. From Eq. (25) it is apparent that $P_N(T_1) \leqslant P_N(T)$ when $T_1 \geqslant T$. Since $P_\phi \leqslant 1$, it then follows that $P_F \leqslant P_N(T)$. It can be shown that $\mathrm{erf}(x) > 1 - 1/2\,\exp(-x^2/2)$ when $x > 0$. Using this inequality and Eq. (26),

$$P_F < \frac{1}{4}\left(1 + \frac{\sigma_n}{\sqrt{\sigma_n^2 + T^2\sigma_{n'}^2}}\right) \exp\left(-\frac{L^2}{2\sigma_n^2}\right), \tag{28}$$

where T is the minimum possible duration of a passed pulse.

PRACTICAL APPLICATION

A situation of common occurrence is a signal consisting of pulses which, in the vicinity of an edge, have the form shown in Fig. 3. The threshold level a is set at half the value of the signal amplitude. In this symmetrical situation the mean crossing time is the time at which the unperturbed signal reaches the threshold level; that is, $s(\bar{t}) = a$. We now seek to evaluate the probability density function given by Eq. (12). As mentioned previously, when $s(t)$ is antisymmetrical about $\bar{t}$, we have $t_1 = \bar{t} - T$ and $t_2 = \bar{t} + T$. The parameter T is determined from Eq. (16).

If $s(t)$ is approximated by Fig. 3, it may be described analytically by

$$\left.\begin{aligned} s(t) &= a + M(t - \bar{t}), && \bar{t} - \frac{a}{M} \leqslant t \leqslant \bar{t} + \frac{a}{M}, \\ s(t) &= 2a, && t > \bar{t} + \frac{a}{M}, \\ s(t) &= 0, && t < \bar{t} - \frac{a}{M} \end{aligned}\right\} \tag{29}$$

where M is the slope of the ramp and is assumed to be a constant; that is, $M = s'(\bar{t})$. It is also assumed that T is large enough that $s(\bar{t} + T) = 2a$; that is, it is assumed that $T > a/M$. We now use Eq. (3), with $\bar{n} = 0$, and Eq. (29) in Eq. (12). The result is

$$
\left.
\begin{aligned}
f_c(t) &= \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left[-\frac{M^2(t-\bar{t})^2}{2\sigma_n^2}\right]\left[M \operatorname{erf}\left(\frac{M}{\sigma_n'}\right) + \frac{\sigma_n'}{\sqrt{2\pi}} \exp\left(-\frac{M^2}{2\sigma_{n'}^2}\right)\right], \\
&\qquad \bar{t} - \frac{a}{M} \leqslant t \leqslant \bar{t} + \frac{a}{M}, \\
f_c(t) &= \frac{\sigma_n'}{2\pi\sigma_n} \exp\left(-\frac{a^2}{2\sigma_n^2}\right) \begin{cases} \bar{t} + \dfrac{a}{M} < t \leqslant \bar{t} + T, \\ \bar{t} - T \leqslant t < \bar{t} - \dfrac{a}{M}, \end{cases} \\
f_c(t) &= 0, \qquad t < \bar{t} - T,\ t > \bar{t} + T.
\end{aligned}
\right\} \tag{30}
$$

This equation may now be combined with Eqs. (16) and (20), which yields

$$
\begin{aligned}
\sigma_c^2 = \Bigg[&\operatorname{erf}\left(\frac{M}{\sigma_n'}\right) + \frac{\sigma_n'}{M\sqrt{2\pi}} \exp\left(-\frac{M^2}{\sigma_{n'}^2}\right)\Bigg]\Bigg[\operatorname{erf}\left(\frac{a}{\sigma_n}\right) - \operatorname{erf}\left(-\frac{a}{\sigma_n}\right) \\
&- \sqrt{\frac{2}{\pi}}\frac{a}{\sigma_n} \exp\left(-\frac{a^2}{2\sigma_n^2}\right)\Bigg]\frac{\sigma_n^2}{M^2} + \left(T^3 - \frac{a^3}{M^3}\right)\frac{\sigma_n'}{3\pi\sigma_n} \exp\left(-\frac{a^2}{2\sigma_n^2}\right),
\end{aligned} \tag{31}
$$

and

$$
\begin{aligned}
T = \frac{a}{M} + \frac{\pi\sigma_n}{\sigma_n'} \exp\left(\frac{a^2}{2\sigma_n^2}\right)\Bigg\{1 - \Bigg[&\operatorname{erf}\left(\frac{M}{\sigma_n'}\right) \\
&+ \frac{\sigma_n'}{M\sqrt{2\pi}} \exp\left(-\frac{M^2}{2\sigma_{n'}^2}\right)\Bigg]\left[\operatorname{erf}\left(\frac{a}{\sigma_n}\right) - \operatorname{erf}\left(-\frac{a}{\sigma_n}\right)\right]\Bigg\}
\end{aligned} \tag{32}
$$

The condition under which Eq. (31) is a good approximation is given by Eq. (22). Noting that $M = 0$ for $t > \bar{t} + (a/M)$ and for $t < \bar{t} - (a/M)$, Eq. (22) becomes

$$
2 \operatorname{erf}\left(-\frac{a}{\sigma_n}\right) \ll 1. \tag{33}
$$

A tangential signal-to-noise ratio is defined as 8.5 dB. For this value $a/\sigma_n \approx 1.33$ and $2 \operatorname{erf}(-a/\sigma_n) \approx 0.18$. Thus the above inequality is barely satisfied when a tangential signal is present. The left-hand side of Eq. (23) represents the probability of a reverse threshold crossing in the interval $\bar{t} - T < t < \bar{t} + T$. Certainly, if this probability is 18%, the theoretical development of the first section becomes highly questionable. Therefore, it may be concluded that the signal-to-noise ratio must be somewhat greater than tangential for Eq. (31) to provide an accurate value of the standard deviation.

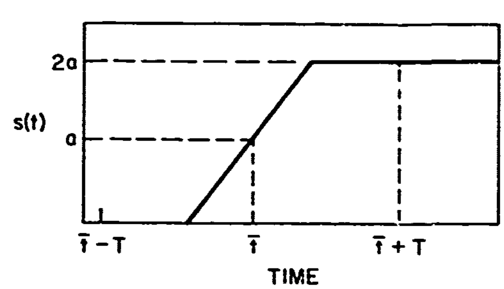

Fig. 3—Edge of an idealized pulse

To use Eq. (31), we must know the values of M, σ_n, and σ_n'. Each of these quantities is a function of the IF filter shown in Fig. 1. In addition the detection process employed by the

threshold detector may alter one or more of these quantities. In the next two sections, we shall consider specific choices of the threshold detector and the IF filter.

PEAK-AMPLITUDE ESTIMATOR

A simplified schematic diagram of a peak amplitude estimator is shown in Fig. 4. This circuit is an adaptive threshold detector; that is, it will locate the leading edge of a pulse regardless of its amplitude. In effect, the threshold level is adjusted when the pulse amplitude changes. The input signal and noise, $s_i(t) + n_i(t)$, are fed into this threshold detector from the demodulator, as shown in Fig. 1. The operation of this circuit is most easily explained if it is initially assumed that $n_i(t) = 0$, that the filter bandwidth is wide, and that the amplifier has a gain $G = -2$. Suppose that $s_i(t)$ has the form indicated in Fig. 5. Referring to Figs. 4 and 5, it is seen that the output signal $s_0(t)$ is a sharp pulse rising at time $\bar{t} + \tau$, where τ is the time delay of the delay line. In Fig. 5, δ is the delay of the filter. It should be noticed that $s_2(t)$ goes from above the threshold to below it, whereas our derivations have assumed the opposite. However, it may be easily verified that a threshold crossing from above to below does not change any of our formulas if we merely interpret M as the absolute value of the slope and a as the absolute value of the threshold level.

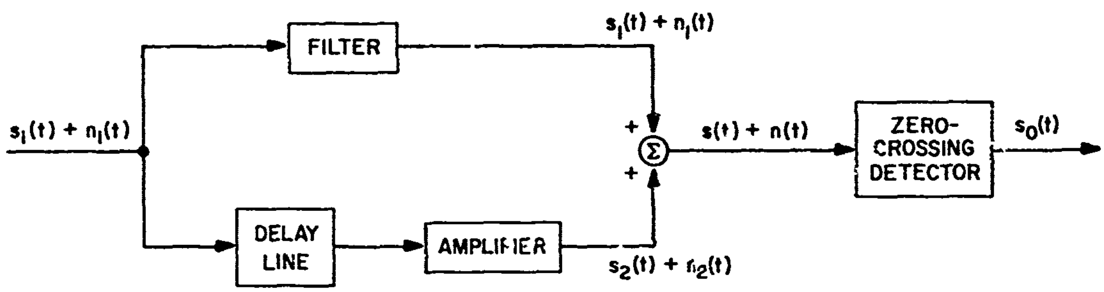

Fig. 4—Peak-amplitude estimator

When noise is present at the detector input, there will be two contributions to the zero-crossing fluctuation of $s(t)$. One contribution is the noise at the leading edge of $s_2(t)$; the other is the amplitude fluctuation of the flat portion of $s_1(t)$. The latter fluctuation is due to the noise in the upper branch of the circuit of Fig. 4.

A filter is inserted to reduce the noise in the upper branch. However, decreasing the filter bandwidth lengthens the rise time of $s_1(t)$. Consequently, the amplitude of $s_1(t)$ at time $\bar{t} + \tau$ is decreased. As seen in Fig. 5, this decrease will cause an erroneous early detection of the pulse. To remedy this situation, the amplifier of the lower branch must have its gain adjusted. If the amplifier gain is G,

$$s(\bar{t} + \tau) = s_1(\bar{t} + \tau) + s_2(\bar{t} + \tau) = s_1(\bar{t} + \tau) + G\, s_i(\bar{t}). \tag{34}$$

To ensure that $s(\bar{t} + \tau) = 0$ in the absence of noise, it is necessary that

$$G = -\frac{s_1(\bar{t} + \tau)}{s_i(\bar{t})}. \tag{35}$$

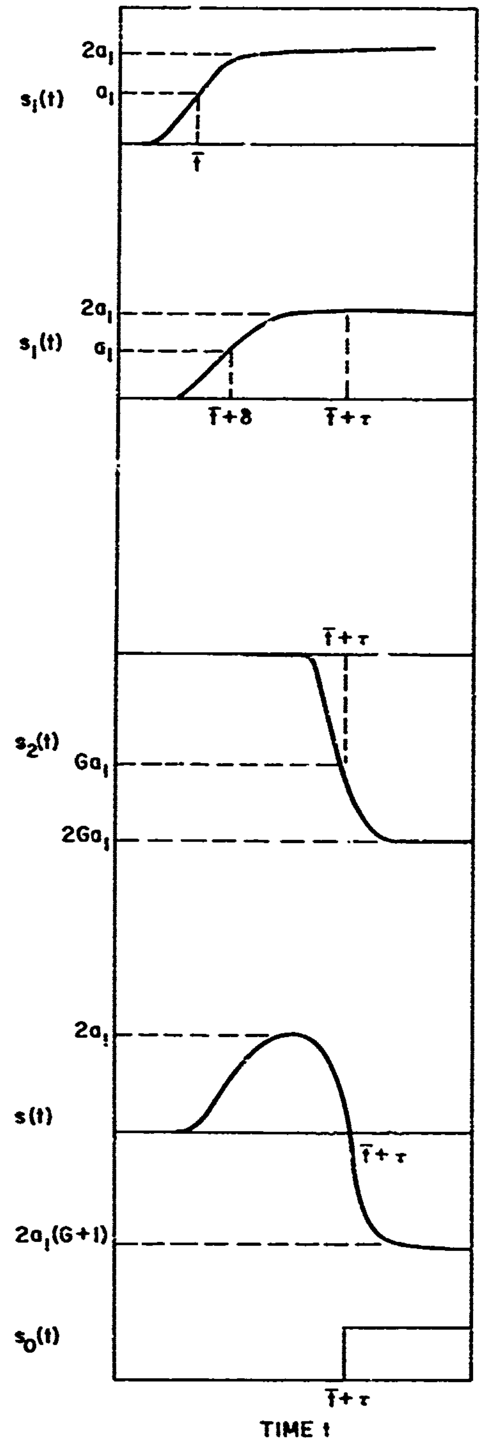

Fig. 5—The signals of Fig. 4

If Eq. (35) is not satisfied, the zero crossing of $s(t)$ will occur at a time $\bar{t} + \tau + \Delta t$. If Δt is small,

$$\Delta t \approx -\frac{s(\bar{t} + \tau)}{s'(\bar{t} + \tau)} = -\frac{s_1(\bar{t} + \tau) + Gs_i(\bar{t})}{s_1'(\bar{t} + \tau) + Gs_i'(\bar{t})}, \tag{36}$$

where the prime denotes differentiation. This change in zero-crossing time is equivalent to a change $\Delta c = s_i'(t)\Delta t$ in the threshold level which defines the pulse-edge location. From now on, we assume that Eq. (35) is satisfied and, consequently, that $\Delta t = 0$.

Since we are interested in determining the position of the edge of a pulse, we can approximate the pulse by a step function to simplify the mathematics. As mentioned previously, the signal entering the threshold detector may be considered the result of a pulse passing through an equivalent baseband filter (Fig. 1). Let $u(t)$ be the signal entering this filter and $h(t)$ be the impulse response. Then

$$s_i(t) = \int_{-\infty}^{\infty} h(t - t')u(t')\,dt', \tag{37}$$

where $s_i(t)$ is the input signal in Fig. 4. Suppose the filter transfer function is $H(\omega) = |H(\omega)|e^{-ik\omega}$; that is, the filter phase response is assumed to be linear over the range of significant values of $|H(\omega)|$. It is also assumed that $H(0) = 1$. If $u(t)$ is a step function of amplitude $2a_1$, occurring at $t = 0$, the convolution theorem may be used with Eq. (37) to show that $s_i(k) = a_1$. Thus we may make the identification $\bar{t} = k$. It then follows from Eq. (37) and the convolution theorem that

$$s_i'(t) = \frac{a_1}{\pi} \int_{-\infty}^{\infty} |H(\omega)|e^{j\omega(t-\bar{t})}\,d\omega. \tag{38}$$

It is immediately noticed that $s_i'(t)$ is maximum at $t = \bar{t}$. Referring to Figs. 4 and 5, the absolute value of the slope of $s(t)$ at $t = \bar{t} + \tau$ is

$$M = |s'(\bar{t} + \tau)| = -s_2'(\bar{t} + \tau) - s_1'(\bar{t} + \tau) = -Gs_i'(\bar{t}) - s_1'(\bar{t} + \tau), \tag{39}$$

where G is assumed to be negative. It is assumed that the delay time τ is chosen large enough that $s_1'(t + \tau) \approx 0$. Then Eqs. (38) and (39) yield

$$M = -G\frac{a_1}{\pi} \int_{-\infty}^{\infty} |H(\omega)|\,d\omega. \tag{40}$$

The mean-square noise associated with $s(t)$ is

$$\sigma_n^2 = \overline{n^2(t)} = \overline{[n_1(t) + n_2(t)]^2} = \overline{n_1^2(t)} + \overline{n_2^2(t)} + \overline{2n_1(t)n_2(t)}. \tag{41}$$

To evaluate the right-hand side of Eq. (41), we let $S_0(\omega)$ denote the power spectrum of the noise entering the IF filter. Then the power spectrum of the noise entering the threshold detector is

$$S_i(\omega) = S_0(\omega)|H(\omega)|^2, \tag{42}$$

where $H(\omega)$ is the transfer function of the IF filter. The delay line has no effect on the power spectrum, so

$$\overline{n_2^2(t)} = \frac{G^2}{2\pi} \int_{-\infty}^{\infty} S_i(\omega)\, d\omega = \frac{G^2}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2\, d\omega. \tag{43}$$

If we let $H_1(\omega)$ represent the transfer function of the filter in the upper branch of Fig. 4, we similarly obtain

$$\overline{n_1^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2|H_1(\omega)|^2\, d\omega. \tag{44}$$

After some manipulation the last term in Eq. (41) becomes

$$\overline{n_1(t)n_2(t)} = \frac{G}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2|H_1(\omega)|e^{j\omega\tau_r}\, d\omega, \tag{45}$$

where τ_r is the relative delay time between the two branches. In general, it is exceedingly difficult to evaluate the integral of Eq. (45). However, if both $H(\omega)$ and $H_1(\omega)$ are Gaussian in shape and if $S_0(\omega)$ has a flat spectrum, then it is found that $\overline{n_1(t)n_2(t)}$ is negligibly small if $\tau_r > B^{-1}$, where B is the equivalent "bandwidth" of $|H(\omega)|^2|H_1(\omega)|$. Henceforth, it will be assumed that $\overline{n_1(t)n_2(t)}$ is negligible. We are left with

$$\sigma_n^2 = \frac{G^2}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2\, d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2|H_1(\omega)|^2\, d\omega. \tag{46}$$

The differentiation of a noise process is equivalent to passing it through a filter with transfer function $H_D(\omega) = j\omega$. Using this fact and calculating as done previously, we obtain for the mean square of $n'(t)$

$$\sigma_{n'}^2 = \frac{G^2}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2\omega^2\, d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2|H_1(\omega)|^2\omega^2\, d\omega. \tag{47}$$

Once the filter functions $|H(\omega)|$ and $|H_1(\omega)|$ are specified, Eqs. (40), (46), and (47) may be used in Eq. (31) to obtain the standard deviation of the threshold crossing time. As a specific example, let

$$|H(\omega)| = \exp\left(-\frac{\omega^2}{2\omega_0^2}\right). \tag{48}$$

If the bandwidth is defined as the frequency difference between the half-power points of $|H(\omega)|$, it is easy to show that the bandwidth B and the parameter ω_0 are related by

$$\omega_0 = \frac{\pi}{\sqrt{\log 2}} B \tag{49}$$

We assume that $H_1(\omega)$ is an RC "integrator" filter; that is,

$$H_1(\omega) = \frac{1}{j\omega RC + 1} \tag{50}$$

It is assumed in this example that the noise is white; that is, $S_0(\omega) = N$, where N is a constant. Using Eqs. (48) and (50) in Eqs. (40), (46), and (47), there results

$$M = -G\sqrt{\frac{2}{\pi}}a_1\omega_0$$

$$\sigma_n^2 = \frac{G^2N}{\sqrt{4\pi}}\omega_0 + \frac{N}{RC}\operatorname{erf}\left(-\frac{\sqrt{2}}{\omega_0 RC}\right)\exp\left[\left(\frac{1}{\omega_0 RC}\right)^2\right] \tag{51}$$

$$\sigma_{n'}^2 = \frac{G^2N}{2\sqrt{4\pi}}\omega_0^3 + \frac{N}{\sqrt{4\pi}(RC)^2}\omega_0 - \frac{N}{(RC)^3}\operatorname{erf}\left(-\frac{\sqrt{2}}{\omega_0 RC}\right)\exp\left[\left(\frac{1}{\omega_0 RC}\right)^2\right]$$

The substitution of Eq. (51) and the identity $a = -Ga_1$ in Eq. (31) yields the standard deviation of the pulse-edge fluctuation.

So far, we have ignored the zero-crossing-time fluctuation caused by the zero-crossing detector itself. Usually, this detector is implemented by a Schmitt-Trigger circuit. Theoretically the Schmitt Trigger will change state when the input voltage crosses the 0-volt level. In reality, triggering may occur anywhere within a voltage range of $2\Delta V$ volts, where ΔV is the uncertainty in the triggering level. If the absolute value of the slope of the detected pulse is M and if t_0 is the mean zero-crossing time, then the detected zero crossing may occur anywhere in the range $t_0 - (\Delta V/M) \leq t \leq t_0 + (\Delta V/M)$. Defining σ_{ST}^2 as the mean-square deviation from the mean zero-crossing time and assuming that there is a uniform probability of a zero crossing anywhere in the above range,

$$\sigma_{ST}^2 = \frac{1}{3}\left(\frac{\Delta V}{M}\right)^2. \tag{52}$$

This equation gives the lower limit of the standard deviation of the time fluctuation of a pulse edge; that is, this equation gives the standard deviation in the total absence of noise and under ideal operating conditions elsewhere in the detector.

DOUBLE DIFFERENTIATOR

It is observed from Eq. (46) that the presence of the upper branch of Fig. 4 causes an increase in the noise power at the zero-crossing detector. It is worthwhile, therefore, to investigate a detection system which contains only a single branch. Such a system is illustrated in Fig. 6, where $s_i(t)$ represents the pulse after passage through the IF filter and $s_i(t)$ is determined by Eq. (37). Once again, the input to the IF filter is assumed to be a step function of amplitude $2a$; the filter transfer function is $H(\omega) = |H(\omega)|e^{-j\omega\bar{t}}$. It follows from Eq. (38) that after passage through the double differentiator, the signal is

$$s(t) = s_i''(t) = \frac{a}{\pi}\int_{-\infty}^{\infty} j\omega|H(\omega)|e^{j\omega(t-\bar{t})}\,d\omega. \tag{53}$$

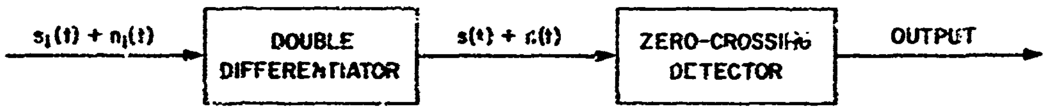

Fig. 6—Double differentiator system

It is seen that $s(t) = 0$ at $t = \bar{t}$, which is the half-amplitude point of $s_i(t)$. Thus the double differentiator has converted the half-amplitude time to the zero-crossing time. When Gaussian noise is present, we may apply Eq. (21) once the values of M, σ_n^2, and $\sigma_{n'}^2$ are known for the system of Fig. 6. From Eq. (53)

$$M = |s'(\bar{t})| = \frac{a}{\pi} \int_{-\infty}^{\infty} \omega^2 |H(\omega)|\, d\omega \tag{54}$$

Since the power spectrum of $n(t)$ is given by $S(\omega) = S_0(\omega)|H(\omega)|^2\omega^4$, it follows that

$$\left.\begin{aligned} \sigma_n^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2 \omega^4\, d\omega, \\ \text{and} \qquad \sigma_{n'}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_0(\omega)|H(\omega)|^2 \omega^6\, d\omega, \end{aligned}\right\} \tag{55}$$

where $S_0(\omega)$ is the power spectrum of the noise entering the IF filter. If the noise is white Gaussian, then $S_0(\omega) = N$, a constant. If the filter transfer function is Gaussian, then Eq. (48) may be applied to Eqs. (54) and (55) with the result that

$$\left.\begin{aligned} M &= \sqrt{\frac{2}{\pi}}\, a\omega_0^3, \\ \sigma_n^2 &= \frac{3N}{8\sqrt{\pi}}\omega_0^5, \\ \text{and} \qquad \sigma_{n'}^2 &= \frac{15N}{16\sqrt{\pi}}\omega_0^7. \end{aligned}\right\} \tag{56}$$

In the general case, a comparison between the double differentiator and the peak-amplitude estimator is difficult without the aid of a computer. However, in the special case of weak noise, we may use Eq. (24) to obtain an approximate comparison. We assume that the signal $s(t)$ has the form shown in Fig. 3 and is described by Eq. (29). We now combine Eqs. (24), (29), (3), and (20). Since $\bar{n} = 0$ and $M = 0$ outside the interval $\bar{t} - (a/M) \leq t \leq \bar{t} + (a/M)$, we calculate

$$\sigma_c^2 = \frac{\sigma_n^2}{M^2}\left[\operatorname{erf}\left(\frac{a}{\sigma_n}\right) - \operatorname{erf}\left(-\frac{a}{\sigma_n}\right) - \frac{2a}{\sqrt{2\pi}\,\sigma_n}\exp\left(-\frac{a^2}{2\sigma_n^2}\right)\right] \tag{57}$$

The threshold value is, as usual, equal to one half the amplitude of the signal. When the noise is weak, the signal-to-noise ratio is high. Thus a/σ_n has a large value, and the bracketed term in Eq. (57) has a value slightly less than unity. We conclude that

$$\sigma_c^2 < \frac{\sigma_n^2}{M^2}. \tag{58}$$

We now evaluate Eq. (58) for the double differentiator. From Eqs. (56) and (58)

$$(\sigma_c^2)_{DD} < \frac{3\sqrt{\pi}N}{16a^2\omega_0}. \tag{59}$$

The corresponding equation for the peak-amplitude estimator is obtained from Eq. (51). We have

$$(\sigma_c^2)_{PA} > \frac{\sqrt{\pi}N}{4a_1^2\omega_0}. \tag{60}$$

To make a fair comparison, we must assume that both detectors are fed by signals of equal amplitude and that both detectors have half-amplitude threshold levels. With this assumption, $a_1 = a$, and Eqs. (59) and (60) may be combined. The result is

$$(\sigma_c^2)_{DD} < \frac{3}{4}(\sigma_c^2)_{PA}. \tag{61}$$

Thus the double differentiator gives a significantly superior performance with respect to the standard deviation of the pulse-edge location. The only drawback is that a satisfactory double differentiator is difficult to implement.

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APPENDIX

Non-Gaussian Noise

The preceding derivations have all assumed that the noise is Gaussian. In this section, the noise is assumed to be weak but not necessarily Gaussian. If the noise is weak, it is intuitively reasonable to assume that there is only one threshold-level crossing in the vicinity of the signal edge. Formally, it is assumed that the probability of a reverse crossing (from above the threshold to below it) is zero.

Let $F_n(x)$ be the probability distribution function of the noise; that is, $F_n(x)$ is equal to the probability that $n(t) \leqslant x$. Once again, a is the value of the threshold level and $s(t)$ represents the signal. Since there can be only one crossing, the probability that no crossing occurred before the time t is equal to $F_n[a - s(t)]$, for if $n(t) + s(t) < a$, then no crossing could have occurred before the time t. Let $F_c(t)$ be the probability distribution function of a crossing; that is, $F_c(t)$ is equal to the probability that a crossing occurred before the time t. From this definition and the preceding statements,

$$1 - F_c(t) = F_n[a - s(t)]. \tag{A1}$$

Differentiation of a distribution function gives a density function. Thus at all points where a derivative is defined, we have the following density function for a threshold crossing:

$$f_c(t) = s'(t)\, f_n[a - s(t)] \tag{A2}$$

In general, this expression pertains to the vicinity of the signal edge, where it has been assumed that only one threshold-level crossing occurs. Using the same normalization procedures as in the first section, Eq. (A2) leads to Eq. (24), where $M = s'(t)$ by definition. Originally, Eq. (24) was derived by assuming a single crossing and Gaussian noise. The present derivation shows that Eq. (24) is valid for any noise statistics, so long as the single-crossing assumption applies.

It is noteworthy that Eq. (12) reduces to Eq. (24) when $\sigma_{n'} \to 0$, which occurs when the noise is weak. This observation is a manifestation of the connection between the single-crossing assumption and weak noise.